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Vortices and polynomials

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Abstract

A number of connections between point vortex dynamics and the properties of complex polynomials or rational functions with roots and/or poles at the vortex positions are reviewed. Classical polynomials, such as the Hermite and Laguerre polynomials, have roots that describe vortex equilibria. Completely stationary vortex configurations with vortices of the same strength but positive or negative orientation are given by zeros of the so-called Adler–Moser polynomials. The geometrical characterization of the location of the stagnation points in a flow produced by an assembly of point vortices is addressed. The 1864 theorem of Siebeck provides a beautiful solution to this problem.

Keywords: Point vortex; Equilibrium; Polynomial; Zero; Stagnation point

Contents

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1. Introduction

The notion of a point vortex is a classical approximation in ideal hydrodynamics of planar flow having been introduced almost 150 years ago in Helmholtz's classical paper on vortex dynamics (von Helmholtz, 1858). While we shall study the point vortex model in its own right in this paper, the reader may be interested to pursue the connection to the two-dimensional Euler equation (Marchioro and Pulvirenti, 1996).

Considering the flow plane to be the complex plane, it is tempting to associate with a set of point vortices a polynomial with roots at the locations of the vortices, and to use this representation to discover properties of the vortex configurations. We call such a polynomial a *generating polynomial* for the vortex configuration. This program has so far not been carried out in full generality—and it may not be possible to do so—but a number of interesting and suggestive results have been obtained. In this paper we review the results achieved to date. There are, of course, many other developments in point vortex dynamics. We direct the reader to the recent monograph by Newton (2001).

The equations of motion for N point vortices with circulations Γ_α at positions z_α, $\alpha = 1, \ldots, N$, are:

$$\overline{\frac{\mathrm{d}z_\alpha}{\mathrm{d}t}} = \frac{1}{2\pi \mathrm{i}} {\sum_{\beta=1}^{N}}' \frac{\Gamma_\beta}{z_\alpha - z_\beta}. \tag{1}$$

In (1) the prime on the sum on the right-hand side indicates omission of the singular term $\beta = \alpha$ and the overline on the left-hand side indicates complex conjugation.

The program of embedding vortex configurations as the roots of polynomials has thus far primarily been applied to situations where the vortex configuration moves without change of shape or size. For example, if a vortex configuration rotates as a rigid body with angular velocity Ω, the left-hand sides in (1) become $-\mathrm{i}\Omega\overline{z}_\alpha$, $\alpha = 1, \ldots, N$. If the configuration translates with complex velocity V, the left-hand sides become $\overline{V}$. If the configuration is stationary—this is possible for certain combinations of the vortex strengths—the left-hand sides in (1) are all zero. In all these cases (1) reduces to a set of algebraic equations for the vortex positions. It is these equations that sometimes have solutions by 'the method of polynomials'. Even within this restricted class of solutions the approach has only been successful in cases where the vortex strengths take on particularly simple values.

2. Identical vortices

If we substitute the *Ansatz* $-\mathrm{i}\Omega\overline{z}_\alpha$ on the left-hand sides of (1), we obtain the following system of algebraic equations:

$$\begin{aligned} \lambda\overline{z}_1 &= \frac{\Gamma_2}{z_1 - z_2} + \frac{\Gamma_3}{z_1 - z_3} + \cdots + \frac{\Gamma_N}{z_1 - z_N}, \\ \lambda\overline{z}_2 &= \frac{\Gamma_1}{z_2 - z_1} + \frac{\Gamma_3}{z_2 - z_3} + \cdots + \frac{\Gamma_N}{z_2 - z_N}, \\ &\cdots \\ \lambda\overline{z}_N &= \frac{\Gamma_1}{z_N - z_1} + \frac{\Gamma_2}{z_N - z_2} + \cdots + \frac{\Gamma_{N-1}}{z_N - z_{N-1}}, \end{aligned} \tag{2}$$

where $\lambda = 2\pi\Omega$.

One might think that since the problem for identical vortices is the most symmetrical, it should be the most accessible to analysis. However to date this has not proven to be the case. The reason, undoubtedly, is the appearance of the complex conjugates of the positions on the left-hand sides of Eqs. (2). In order to make progress one also needs the configuration to have a high degree of geometrical symmetry. In Section 2.1 we give a classical example of this kind: the vortices are all situated on a line! In Section 2.2 we review progress and outstanding issues regarding configurations of identical vortices of greater generality. Some analytical progress has been made for configurations of vortices arranged on nested, regular polygons. In addition, considerable 'empirical' evidence has been gathered by direct numerical solution of Eqs. (2) with all Γ_α set equal to Γ.

2.1. Identical vortices on a line

There is a surprising connection between the problem of how to place N identical vortices on a line such that the configuration rotates like a rigid body and the zeros of a well-known family of orthogonal polynomials. The connection was first noted by Stieltjes (1885) who considered interacting line charges. His work and subsequent developments are described by Marden (1949) and by Szegö (1959). This solution has been rediscovered and used many times. For example, Eshelby et al. (1951) used it to model dislocation pile-up.

If in Eqs. (2) we assume that the N vortices are identical and placed on a line, for convenience taken as the x-axis of coordinates, their positions along the line may be designated $x_1, \ldots, x_N$, and Eqs. (2) become

$$\begin{aligned}
\Lambda x_1 &= \frac{1}{x_1 - x_2} + \frac{1}{x_1 - x_3} + \cdots + \frac{1}{x_1 - x_N},\\
\Lambda x_2 &= \frac{1}{x_2 - x_1} + \frac{1}{x_2 - x_3} + \cdots + \frac{1}{x_2 - x_N},\\
&\cdots\\
\Lambda x_N &= \frac{1}{x_N - x_1} + \frac{1}{x_N - x_2} + \cdots + \frac{1}{x_N - x_{N-1}},
\end{aligned} \tag{3}$$

where in physical units $\Lambda = 2\pi\Omega/\Gamma$ with Ω the angular frequency of rotation and Γ the common circulation of the vortices. It is clear that vortices with positive circulation can only rotate with Ω positive, so Λ will be positive. We can then scale all the x's by $\sqrt{\Lambda}$. In other words, it suffices to consider Eqs. (3) with $\Lambda = 1$.

Now embed $x_1, \ldots, x_N$ as roots of the polynomial

$$P(x) = (x - x_1)(x - x_2)\cdots(x - x_N). \tag{4}$$

This polynomial satisfies an ODE of second order which is obtained as follows. First, from its definition, Eq. (4), we see that the derivative of P is

$$P' = P \sum_{\alpha=1}^{N} \frac{1}{x - x_\alpha}. \tag{5}$$

A second differentiation gives

$$P'' = P \sideset{}{'}\sum_{\alpha,\beta=1}^{N} \frac{1}{x - x_\alpha}\frac{1}{x - x_\beta}. \tag{6}$$

Since $x_\alpha \neq x_\beta$ the summand can be re-written:

$$\frac{1}{x-x_\alpha}\frac{1}{x-x_\beta}=\left[\frac{1}{x-x_\alpha}-\frac{1}{x-x_\beta}\right]\frac{1}{x_\alpha-x_\beta}. \tag{7}$$

In the double sum in (6) we then get, according to (3) with $\Lambda=1$,

$$\sum_{\alpha,\beta=1}^{N}{}'\frac{1}{x-x_\alpha}\frac{1}{x-x_\beta}=2\sum_{\alpha,\beta=1}^{N}{}'\frac{1}{x-x_\alpha}\frac{1}{x_\alpha-x_\beta}=2\sum_{\alpha=1}^{N}\frac{x_\alpha}{x-x_\alpha}.$$

Thus,

$$P''=2P\sum_{\alpha=1}^{N}\frac{x_\alpha}{x-x_\alpha}=-2NP+2xP'. \tag{8}$$

We recognize this equation as the differential equation satisfied by the Nth *Hermite polynomial* $H_N(x)$. Since the Hermite polynomial is the unique polynomial solution to this second order ODE, we have established that the solutions to (3) with $\Lambda=1$ are the roots of the Nth Hermite polynomial.

This result is intriguing because it suggests a link between point vortex dynamics and other areas of applied mathematics with which the subject a priori would seem to have no connection whatsoever. This theme will be amplified by the examples given later. The result may be seen as somewhat disappointing because, of course, we have accomplished little in terms of actually finding solutions to our problem—we have simply related one set of unknowns to another! It is a matter of taste whether one feels more information is conveyed by saying that the vortex positions satisfy (3) or that they are roots of H_N. The bigger question, however, is whether the idea of a generating function for the vortex positions, such as the polynomial $P(x)$ introduced in (4), that will satisfy a relatively simple differential equation, carries further. This notion of generating functions has proven extremely powerful in other areas of mathematics, for example in combinatorics, and it would be very interesting if $P(x)$, or some generalization thereof, satisfied an ODE or PDE that allowed non-trivial results to be obtained concerning vortex motion. This is the main idea behind this line of investigation.

To help with any disappointment, we may note that systems such as (3) yield various *sum rules* when one takes moments of them of various orders. To state this in the most general way, consider again the problem in Eq. (2). If we multiply (2) by Γ_α, line by line, and sum, we immediately see that

$$\sum_{\alpha=1}^{N}\Gamma_\alpha z_\alpha=0. \tag{9}$$

The sum on the left-hand side is known as the *linear impulse* of the configuration.

Similarly, if we multiply (2) by $\Gamma_\alpha z_\alpha$, line by line, and sum, a short calculation gives

$$\Omega\sum_{\alpha=1}^{N}\Gamma_\alpha|z_\alpha|^2=\frac{1}{4\pi}\left[\left(\sum_{\alpha=1}^{N}\Gamma_\alpha\right)^2-\sum_{\alpha=1}^{N}\Gamma_\alpha^2\right]. \tag{10}$$

The sum on the left-hand side, that looks like a moment of inertia for the point vortices, is known as the *angular impulse* of the configuration.

Thus, for identical vortices on a line, and for units chosen such that $2\pi\Omega = \Gamma$, we have the sum rule for the squares of the roots of the Nth Hermite polynomial:

$$\sum_{\alpha=1}^{N} x_\alpha^2 = \frac{1}{2} N(N-1). \tag{11}$$

This result is known independently but probably not as widely as more elementary properties of the Hermite polynomials. It is pleasing to obtain this result as a corollary of the correspondence with the relative equilibrium configuration of collinear point vortices.

Of course, (9) also leads to a sum rule, viz that the sum of the roots of the Nth Hermite polynomial is zero. This is well-known since the roots are situated symmetrically about the origin and for odd N zero itself is a root.

It is possible to generalize the results just obtained somewhat. For odd $N = 2n+1$ there will always be a vortex at the origin, and one can consider that this vortex might have a different circulation from the other $N-1$. If the central vortex has circulation $p\Gamma$, where Γ is the common value of the circulations of the remaining vortices, then the positions of the vortices are given, up to a scaling factor, by the roots of the *Laguerre polynomial* $L_n^{(p-1/2)}(x^2)$. For $p = 1$ we return to the case of N identical vortices and, indeed, $H_{2n+1}(x)$ is proportional to $L_n^{(1/2)}(x^2)$. For $p = 0$ we return to the case of $2n$ identical vortices and $H_{2n}(x)$ is known to be proportional to $L_n^{(-1/2)}(x^2)$. See Aref (1995) for more discussion.

As N increases it is known that the roots of the Nth Hermite polynomial become more and more uniformly spaced. One would expect this from the connection with the vortex configurations, since in the limit $N \to \infty$ the collinear equilibria should converge to the infinite line of equally spaced vortices, a time-honored model of a vortex sheet. In the limit one can again consider one vortex to have a different circulation, $p\Gamma$, from the rest. The vortices are then, of course, not equally spaced and one can view this as the problem of finding the equilibrium spacing of a row of vortices with an 'inhomogeneity'. The pleasing result is that the vortex positions are given as the zeros of the *Bessel function* $J_{p-1/2}(x)$.

It is quite remarkable that the linearized stability analysis for these configurations can be carried out analytically to a large extent. We shall not elaborate further on this here, since a rather accessible account exists (Aref, 1995). Many of the underlying mathematical results were obtained by Calogero and co-workers in the 1970s. The monograph by Calogero (2001) is a valuable introduction to this work.

2.2. *More general configurations of identical vortices*

We now turn to the more general problem and summarize some of the results that are known analytically and 'empirically'—by which we mean in this instance via direct numerical solution of Eqs. (2).

First, if we follow the procedure above for a general configuration, we obtain for the generating polynomial $P(z) = (z - z_1)\cdots(z - z_N)$ this generalization of Eq. (8)

$$P'' - 2zP' + 2NP = -4\mathrm{i}P \sum_{\alpha=1}^{N} \frac{y_\alpha}{z - z_\alpha}. \tag{12}$$

For vortices on the x-axis the y_α all vanish and we reproduce (8).

2.2.1. Vortex polygons

If the vortices are arranged at the vertices of a regular N-gon of radius R, we can rotate the coordinates such that $P(z) = z^N - R^N$. The sum on the right-hand side of (12) is

$$\sum_{\alpha=1}^{N} \frac{y_\alpha}{z - z_\alpha} = \sum_{\alpha=1}^{N} \frac{R \sin((2\pi/N)\alpha)}{z - R\mathrm{e}^{\mathrm{i}2\pi/N\alpha}} = \frac{1}{2\mathrm{i}} \sum_{\alpha=1}^{N} \frac{R\mathrm{e}^{\mathrm{i}(2\pi/N)\alpha} - R\mathrm{e}^{-\mathrm{i}(2\pi/N)\alpha}}{z - R\mathrm{e}^{\mathrm{i}(2\pi/N)\alpha}}.$$

This may be transformed as follows:

$$\begin{aligned}
&\frac{1}{2\mathrm{i}} \sum_{\alpha=1}^{N} \left[-1 + \frac{z}{z - R\mathrm{e}^{\mathrm{i}(2\pi\alpha/N)}} - R^2 \frac{1}{R\mathrm{e}^{\mathrm{i}(2\pi\alpha/N)}(z - R\mathrm{e}^{\mathrm{i}(2\pi/N)\alpha})} \right] \\
&\quad = \frac{1}{2\mathrm{i}} \left(-N + \frac{N}{1 - (R/z)^N} - \frac{R^2}{z} \sum_{\alpha=1}^{N} \left[\frac{1}{R\mathrm{e}^{\mathrm{i}(2\pi\alpha/N)}} + \frac{1}{z - R\mathrm{e}^{\mathrm{i}(2\pi\alpha/N)}} \right] \right).
\end{aligned}$$

So, finally,

$$-4\mathrm{i}P \sum_{\alpha=1}^{N} \frac{y_\alpha}{z - z_\alpha} = 2N \left(1 - z^{N-2} \frac{z^2 - R^2}{z^N - R^N} \right) P. \tag{13}$$

In order to satisfy (12) we now have

$$N(N-1)z^{N-2} - 2Nz^N + 2Nz^{N-2}(z^2 - R^2) = 0,$$

or

$$R^2 = \frac{N-1}{2}. \tag{14}$$

For a centered, regular N-gon we would have $P(z) = z(z^N - R^N)$. The calculation of the right-hand side in (12) proceeds as before since the vortex at the origin does not contribute to the sum (but the term $2NP$ on the left-hand side becomes $2(N+1)P$ since there are now $N+1$ vortices). A calculation that parallels the one given above now yields

$$R^2 = \frac{N+1}{2}, \tag{15}$$

where $N+1$ is the total number of vortices in the configuration.

If the configuration consists of several nested, regular polygons, say, s rings with n_p vortices in the pth ring, $p = 1, \ldots, s$, the polynomial P takes the form

$$P(z) = (z^{n_1} - (R_1\mathrm{e}^{\mathrm{i}\varphi_1})^{n_1}) \cdots (z^{n_s} - (R_s\mathrm{e}^{\mathrm{i}\varphi_s})^{n_s}). \tag{16}$$

We could still assume that the coordinates are rotated such that for one of the polynomials the angle $\varphi_p = 0$. However, we prefer to retain the symmetry of the formulae.

The calculation of the right-hand side of (12) now proceeds as follows:

$$\sum_{p=1}^{s}\sum_{\alpha=1}^{n_p}\frac{y_\alpha}{z-z_\alpha} = \sum_{p=1}^{s}\sum_{\alpha=1}^{n_p}\frac{R_p\sin(2\pi\alpha/n_p+\varphi_p)}{z-R_p\mathrm{e}^{\mathrm{i}(2\pi\alpha/n_p+\varphi_p)}}$$
$$= \frac{1}{2\mathrm{i}}\sum_{p=1}^{s}\sum_{\alpha=1}^{n_p}\frac{R_p\mathrm{e}^{\mathrm{i}(2\pi\alpha/n_p+\varphi_p)}-R_p\mathrm{e}^{-\mathrm{i}(2\pi\alpha/n_p+\varphi_p)}}{z-R_p\mathrm{e}^{\mathrm{i}(2\pi\alpha/n_p+\varphi_p)}}$$
$$= \frac{1}{2\mathrm{i}}\sum_{p=1}^{s} n_p\left(-1+z^{n_p-2}\frac{z^2-R_p^2}{z^{n_p}-(R_p\mathrm{e}^{\mathrm{i}\varphi_p})^{n_p}}\right). \tag{17}$$

Let us introduce the abbreviation

$$\mu_p(z) = z^{n_p}-(R_p\mathrm{e}^{\mathrm{i}\varphi_p})^{n_p}, \tag{18}$$

such that

$$P(z) = \mu_1\mu_2\cdots\mu_s.$$

Then

$$P' = P\sum_{p=1}^{s}\frac{\mu_p'}{\mu_p}, \quad P'' = P\left[{\sum_{p,q=1}^{s}}'\frac{\mu_p'\mu_q'}{\mu_p\mu_q}+\sum_{p=1}^{s}\frac{\mu_p''}{\mu_p}\right],$$

where the prime on the double sum means $p \neq q$.

The functions v_p,

$$v_p(z) = \frac{n_p}{1-(zR_p\mathrm{e}^{\mathrm{i}\varphi_p})^{n_p}}, \tag{19}$$

introduced by Aref and van Buren (2005), are related to the μ_p through

$$\mu_p(z) = \frac{n_p z^{n_p}}{v_p(1/z)}. \tag{20}$$

Using (18) in the numerator and (20) in the denominator we have

$$\frac{\mu_p'(z)}{\mu_p(z)} = \frac{1}{z}v_p\left(\frac{1}{z}\right), \quad \frac{\mu_p''}{\mu_p} = (n_p-1)\frac{1}{z^2}v_p\left(\frac{1}{z}\right). \tag{21}$$

From (17) we now have

$$\sum_{p=1}^{s}\sum_{\alpha=1}^{n_p}\frac{y_\alpha}{z-z_\alpha} = \frac{1}{2\mathrm{i}}\left[-N+\sum_{p=1}^{s}\left(1-\frac{R_p^2}{z^2}\right)v_p\left(\frac{1}{z}\right)\right]. \tag{22}$$

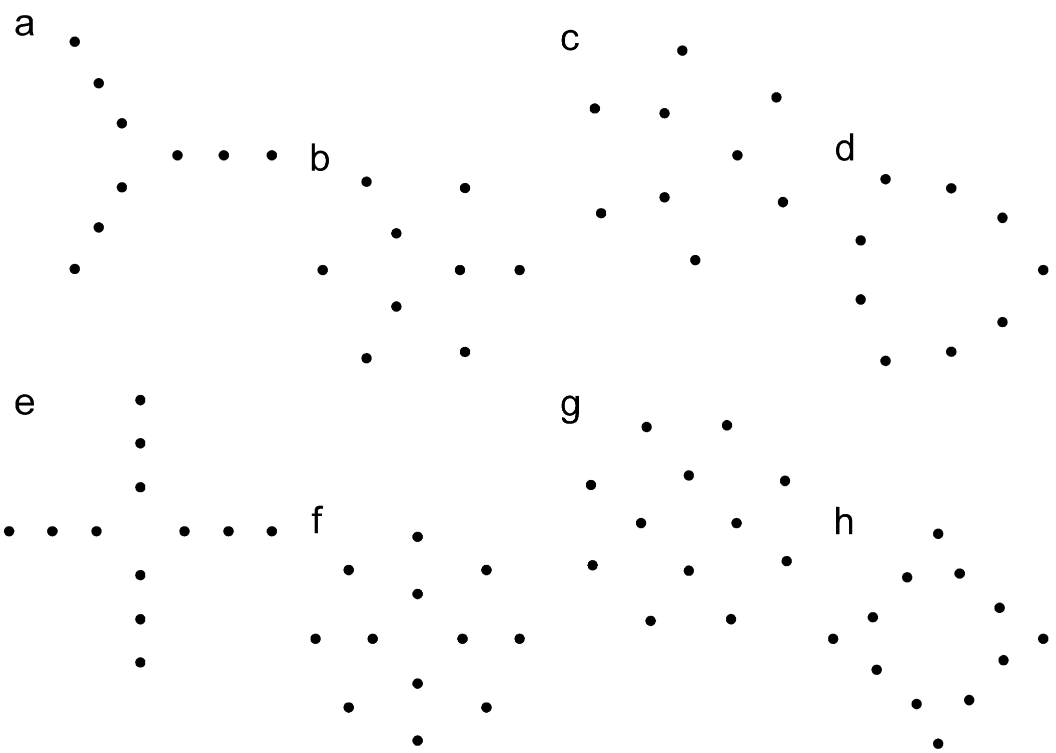

Fig. 1. Examples of equilibria of identical vortices consisting of nested, regular n-gons for $n = 3, 4$. Panels (a) and (e) show the unique symmetric configuration; (b) and (f) the unique staggered configuration; (c), (d), (g) and (h) show 'degenerate' cases where two of the n-gons are of the same size but do not form a regular $2n$-gon.

Thus,

$$P'' - 2zP' + 2NP = \frac{1}{z^2} P \left[{\sum_{p,q=1}^{s}}' v_p v_q + \sum_{p=1}^{s} (n_p - 1) v_p \right] - 2P \sum_{p=1}^{s} v_p + 2NP = -2P \left[-N + \sum_{p=1}^{s} \left(1 - \frac{R_p^2}{z^2} \right) v_p \right],$$

where all the v_p are evaluated at $1/z$. Effecting the obvious cancellations, this gives

$$\sum_{p=1}^{s} (2R_p^2 - n_p + 1) v_p = {\sum_{p,q=1}^{s}}' v_p v_q, \tag{23}$$

which is the main equation used by Aref and van Buren (2005) to discuss equilibrium configurations with several nested, regular polygons, see Fig. 1.

We mention as an aside that the case of two nested, regular n-gons leads to some interesting symmetric polynomials that the ratio of the radii of the two polygons, ξ, must satisfy. For the case of symmetrically

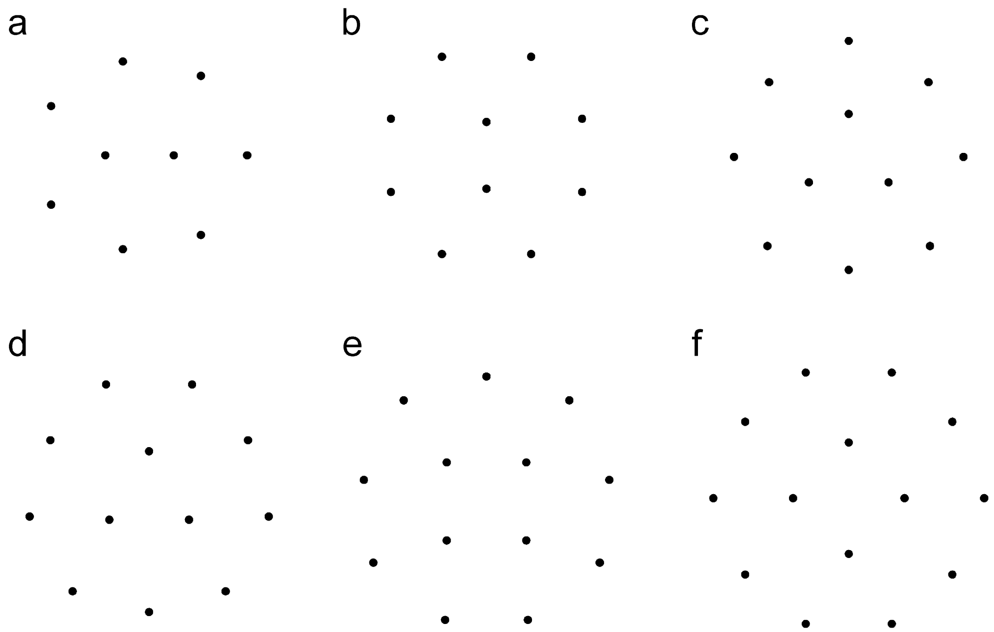

Fig. 2. Equilibria of identical vortices found by direct numerical solution of Eq. (24). From the *Los Alamos Catalog* of Campbell and Ziff (1978) (see also (Campbell and Ziff, 1979)). The solutions shown are (a) 9_1, (b) 10_1, (c) 11_1, (d) 12_1, (e) 13_1, (f) 14_1 in the labeling of the *Catalog*. All are linearly stable to small perturbations.

arranged n-gons the relevant polynomial equation is

$$(n-1)\xi^{n+2} - (3n-1)\xi^n - (3n-1)\xi^2 + n - 1 = 0.$$

If the polygons are staggered, the equation to be solved is

$$(n-1)\xi^{n+2} - (3n-1)\xi^n + (3n-1)\xi^2 - (n-1) = 0.$$

For two n-gons with a vortex at the center we get in the symmetrical case

$$(n+1)\xi^{n+2} - (3n+1)\xi^n - (3n+1)\xi^2 + n + 1 = 0,$$

and in the staggered case

$$(n+1)\xi^{n+2} - (3n+1)\xi^n + (3n+1)\xi^2 - (n+1) = 0.$$

2.2.2. *Symmetry considerations*

The configurations discussed so far have had a high degree of geometrical symmetry. The basic equations we are discussing are, after rescaling,

$$\bar{z}_\alpha = \sum_{\beta=1}^{N}{}' \frac{1}{z_\alpha - z_\beta}. \tag{24}$$

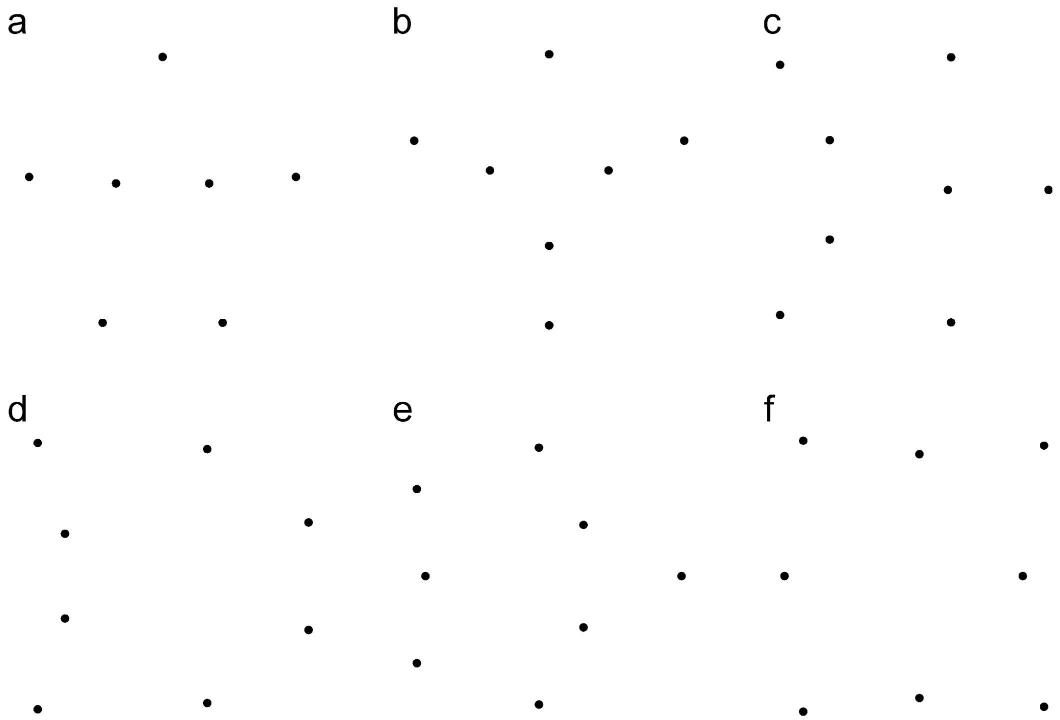

Fig. 3. Examples of equilibria of identical vortices found by direct numerical solution of Eq. (24). All these equilibria have an axis of symmetry but are less 'well-rounded' than the configurations in Fig. 2. After Aref and Vainchtein (1998).

They are invariant to two symmetry transformations: an arbitrary rotation about the origin and complex conjugation (i.e., reflection in the real axis). One might assume from this that solutions to these equations would either have some level of rotational symmetry or, at least, have an axis of symmetry (which could be taken to coincide with the real axis).

Indeed, the majority of known equilibrium configurations have one or both of these symmetries. The nested polygons just discussed are examples of configurations with rotational symmetry. It is interesting to note that a large number of known configurations have just an axis of symmetry. At first this seems counterintuitive. However, it may help to recall that for patches of uniform vorticity we have both the Rankine vortex, which is a circle of uniform vorticity, *and* the Kirchhoff ellipse, as the name suggests an ellipse of uniform vorticity. Both are steadily rotating solutions of the two-dimensional Euler equation. (Because of the jump in vorticity at the boundary we would, strictly speaking, call them weak solutions.) Thus, the notion that a steadily rotating distribution of vorticity has an axis of symmetry is not entirely foreign. We have some analytical understanding of the configurations with rotational symmetry through work of the kind summarized in Section 2.2.1. We have rather limited analytical understanding of the configurations that have only an axis of symmetry. In terms of a generating polynomial $P(z)=(z-z_1)(z-z_2)\cdots(z-z_N)$, the axis of symmetry implies that we can assume all the coefficients in the polynomial to be real since the roots are either real or come in complex conjugate pairs.

For a number of years it was an unstated conjecture that all equilibria of identical vortices had an axis of symmetry. This conjecture was based, in part, on 'empirical' evidence from an early comprehensive

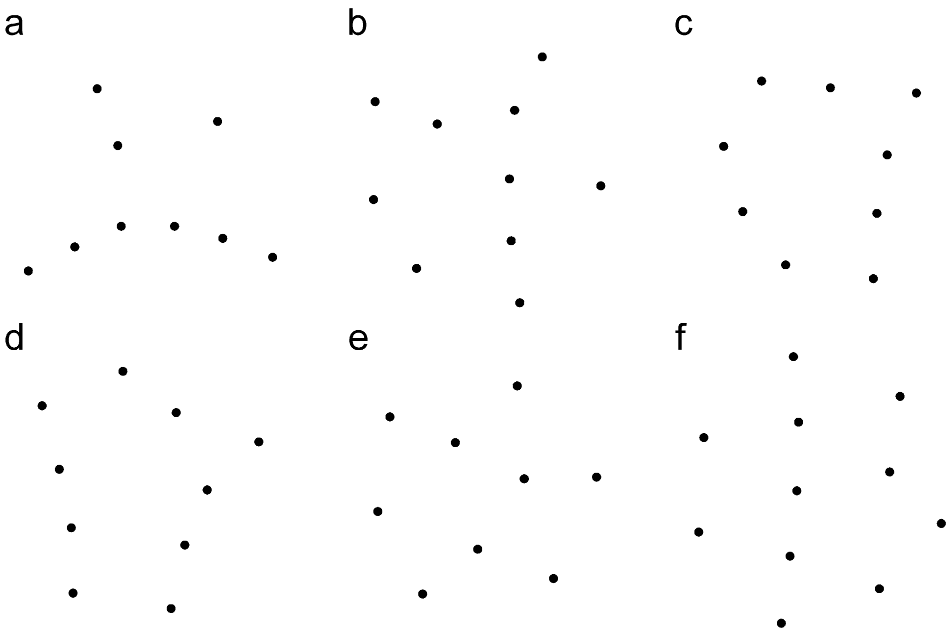

Fig. 4. Examples of asymmetric equilibria of identical vortices found by direct numerical solution of Eq. (24). After Aref and Vainchtein (1998): (a) 9 vortices, (b) 10, (c) 9, (d) 10, (e) 9, (f) 11. Heuristically, configurations (c) and (d) appear 'related', as do (b) and (f).

numerical study by Campbell and Ziff (1978). This report was widely circulated but was never published in the archival literature. A companion paper, Campbell and Ziff (1979), contains several of the results and may be more easily accessible. In the *Los Alamos Catalog*, as the study by Campbell and Ziff (1978) is called, all the configurations found have at least an axis of symmetry. The *Los Alamos Catalog* also contains several of the nested polygon configurations that we have just discussed. The claim is made that the *Catalog* determines all *linearly stable* configurations for $1 < N \leqslant 30$ identical vortices, a claim that has thus far stood the test of time (Fig. 2).

By solving Eq. (24) using a new numerical approach Aref and Vainchtein (1998) found a number of configurations that give a very different visual impression from the 'well-rounded' configurations in Fig. 3. These configurations do all have an axis of symmetry. A separate calculation shows that the configurations in Fig. 3 are all linearly unstable. The space of solutions of Eq. (24) is thus quite rich and varied. Nevertheless, the conjecture that all equilibria have at least an axis of symmetry still appeared to hold. We have limited analytical understanding of the states shown in Fig. 3 since some simplification of Eq. (12) arises because of the existence of the symmetry axis.

However, the study by Aref and Vainchtein (1998) also found a number of equilibria that lack both rotational and reflectional symmetry. In fact, these states have no discernible symmetry at all. On the basis of numerical explorations we believe that such states exist for all $N \geqslant 8$ and that the number of asymmetric equilibria increases with N. We currently have no analytical understanding of these states whatsoever. Note that each panel of Fig. 4 really corresponds to two distinct configurations since the reflection is also a solution of the equations and will not arise from the state shown by rotation. These states invalidate the conjecture that there must at least exist an axis of symmetry for solutions of Eq. (24). Unfortunately, the existence of these asymmetric solutions also suggests that the 'method of polynomials' is likely to become rather complicated if and when it is carried through. At least, the polynomials in question are not confined to those with real coefficients (which would tend to rule out most families of 'known' polynomials).

3. Vortices with the same absolute strength but mixed signs

Another instance where it has been productive to use 'the method of polynomials' is for systems of point vortices with the same absolute strengths but opposite signs, i.e., Γ_α is either $+\Gamma$ or $-\Gamma$ for each vortex, $\alpha = 1, 2, \ldots, N$. The reason is that for such systems configurations are possible that either translate uniformly (this requires equal numbers of positive and negative vortices) or are stationary (the necessary conditions for this are developed below). Hence, in Eq. (2) we have the same complex number (which for the stationary equilibria is zero) on the left-hand sides and the "inconvenience" of the complex conjugates disappears. In this section we review the very interesting and beautiful theory for stationary equilibria with vortices of the same absolute strength.[1]

It may sound surprising that configurations of point vortices exist where the total circulation is non-zero, yet every vortex in the pattern is stationary. A single vortex, of course, has precisely this property. Furthermore, when the right-hand side in Eq. (10) is zero but the angular impulse is non-zero, it follows that Ω must vanish and the configuration is stationary. For example, an equilateral triangle of identical vortices with a vortex of opposite sign at its center satisfies this condition and is an example of a stationary configuration.

When the vortex strengths are "quantized", as indicated above, the condition that the right-hand side of (10) vanishes amounts to saying that

$$(N_+ - N_-)^2 = N_+ + N_-,$$

where N_+ is the number of positive vortices and N_- the number of negative vortices. The total number of vortices N is, of course, $N_+ + N_-$. Set $N_+ - N_- = n$, and the above equation reads $N = N_+ + N_- = n^2$. Now solve these two equations together to obtain

$$N_- = \tfrac{1}{2}n(n-1), \quad N_+ = \tfrac{1}{2}n(n+1), \quad n = 1, 2, \ldots \,. \tag{25}$$

Thus, we have shown as a necessary condition that the number of vortices of the two circulations must be successive triangular numbers (and we have, arbitrarily, chosen the majority population to be the positive-circulation vortices). The total number of vortices is a square. This counting captures the single vortex, which formally constitutes the smallest vortex 'system' with the square of the sum of circulations being equal to the sum of the squares. The centered equilateral triangle will turn out to be the unique example for $n = 2$.

For general N we set

$$P(z) = (z - z_1) \cdots (z - z_{N_+}), \quad Q(z) = (z - \zeta_1) \cdots (z - \zeta_{N_-}). \tag{26}$$

[1] This material closely follows Section VIII of Aref et al. (2002).

Here $z_1, \ldots, z_{N_+}$ are the complex positions of the N_+ positive vortices, whereas $\zeta_1, \ldots, \zeta_{N_-}$ are the positions of the N_- negative vortices, where N_- and N_+ are as in (25). Setting the left-hand sides in Eq. (2) to zero, the equations determining these positions read:

$$\sum_{\beta=1}^{N_+}{}' \frac{1}{z_\alpha - z_\beta} = \sum_{\lambda=1}^{N_-} \frac{1}{z_\alpha - \zeta_\lambda}, \tag{27}$$

$$\sum_{\alpha=1}^{N_+} \frac{1}{\zeta_\lambda - z_\alpha} = \sum_{\mu=1}^{N_-}{}' \frac{1}{\zeta_\lambda - \zeta_\mu}, \tag{28}$$

where $\alpha = 1, \ldots, N_+$, $\lambda = 1, \ldots, N_-$.

Now calculate as before

$$P'(z) = P(z) \sum_{\alpha=1}^{N_+} \frac{1}{z - z_\alpha}, \qquad Q'(z) = Q(z) \sum_{\lambda=1}^{N_-} \frac{1}{z - \zeta_\lambda},$$

$$P''(z) = P(z) \sum_{\alpha,\beta=1}^{N_+}{}' \frac{1}{z - z_\alpha}\frac{1}{z - z_\beta}, \qquad Q''(z) = Q(z) \sum_{\lambda,\mu=1}^{N_-}{}' \frac{1}{z - \zeta_\lambda}\frac{1}{z - \zeta_\mu},$$

$$P''(z) = 2P(z) \sum_{\alpha,\beta=1}^{N_+}{}' \frac{1}{z - z_\alpha}\frac{1}{z_\alpha - z_\beta}, \qquad Q''(z) = 2Q(z) \sum_{\lambda,\mu=1}^{N_-}{}' \frac{1}{z - \zeta_\lambda}\frac{1}{\zeta_\lambda - \zeta_\mu}.$$

At this point use (27) and (28) to re-write $P''(z)$ and $Q''(z)$ as

$$P''(z) = 2P(z) \sum_{\alpha=1}^{N_+} \frac{1}{z - z_\alpha} \sum_{\lambda=1}^{N_-} \frac{1}{z_\alpha - \zeta_\lambda},$$

and

$$Q''(z) = 2Q(z) \sum_{\lambda=1}^{N_-} \frac{1}{z - \zeta_\lambda} \sum_{\alpha=1}^{N_+} \frac{1}{\zeta_\lambda - z_\alpha},$$

respectively.

From these relations and those above one easily obtains

$$QP'' + PQ'' = 2P'Q'. \tag{29}$$

We have called this result *Tkachenko's equation*, since it was first derived by Tkachenko in 1964.

Consider by way of example the case $n = 2$. We have

$$P(z) = z^3 + az^2 + bz + c, \qquad Q(z) = z + d,$$

where a, b, c and d are constant coefficients to be determined. Substitution in (29) gives

$$(6z + 2a)(z + d) = 6z^2 + (2a + 6d)z + 2ad = 2(3z^2 + 2az + b),$$

i.e., $a=3d$ and $ad=b$ or $b=3d^2$. Now, since the square of the sum of the three strengths equals the sum of their squares and, thus, is non-zero, we may always shift our coordinates such that $z_1+z_2+z_3-\zeta_1=0$. Assume this is done. Then $a=-(z_1+z_2+z_3)=-\zeta_1=d$, which together with the results just stated implies $a=d=0$. Then $b=0$, and the unique solution to Tkachenko's (29) for $n=2$ has the form $P(z)=z^3+c$, $Q(z)=z$. This represents three positive vortices in an equilateral triangle with the negative vortex at its center.

This 'method of polynomials' shows its strength as we go to larger n. We will only sketch the argument here since a more detailed exposition is available in Aref et al. (2002).

First, one shows the following result: if P, Q are polynomials (not identically zero) that satisfy Tkachenko's (29), and R is a polynomial that satisfies

$$R'Q - RQ' = P^2, \tag{30}$$

then P and R satisfy (29). The advantage of this is that (30) is a first order differential equation for R (given P and Q), whereas (29) is a second order differential equation for P given Q. This permits the polynomials for successive n to be found recursively.

To set up the recursion, we consider a sequence of polynomials, P_n, generated as follows:

$$P'_{n+1}P_{n-1} - P'_{n-1}P_{n+1} = (2n+1)P_n^2, \quad n=0,1,2,\ldots \tag{31}$$

starting from $P_0=1$, $P_1=z$. For $n=1$ the recursion formula (31) with these starting polynomials tells us that

$$P'_2 = 3z^2,$$

i.e., $P_2=z^3+\text{const.}$, so the start of the recursive construction is in order.

Assume we have constructed polynomials through P_n recursively from (31). We wish to verify that P_{n+1}, constructed from (31), and P_n satisfy (29). We also verify that the normalization factor, $2n+1$, on the right-hand side of (31) leads to the coefficient of the highest power in P_n being 1.

We first use the result (30) with $Q=P_{n-1}$, $P=\sqrt{2n+1}\,P_n$, and $R=P_{n+1}$. By assumption P_{n-1} and P_n, or equivalently Q and P, satisfy Tkachenko's equation (29). Since Q, P and R are to satisfy (30), we conclude that P and R, or equivalently P_n and P_{n+1}, will satisfy (29).

Next, we assume that P_{n-1} and P_n both have 1 as the coefficient of their highest order terms. If the degree of P_n is denoted d_n, equating terms of highest degree in the recursion formula (both matching the degree itself and considering the coefficient) tells us that

$$d_{n+1} - d_{n-1} = 2n+1,$$

$$d_{n+1} - 1 + d_{n-1} = 2d_n.$$

These relations, which are equivalent to $d_{n+1}=d_n+n+1$, show that $d_n=n(n+1)/2$, as expected, and also that (31) assures 1 as the coefficient of the highest order term in each of the polynomials P_n.

All this is straightforward. It takes more thought and work to see that (31), viewed as a first order ODE for P_{n+1}, with P_{n-1} and P_n already determined polynomials of the appropriate degrees, will again yield a polynomial. An elementary argument to this effect appears to have been given already by Burchnall and Chaundy (1929). We shall not repeat this argument here.

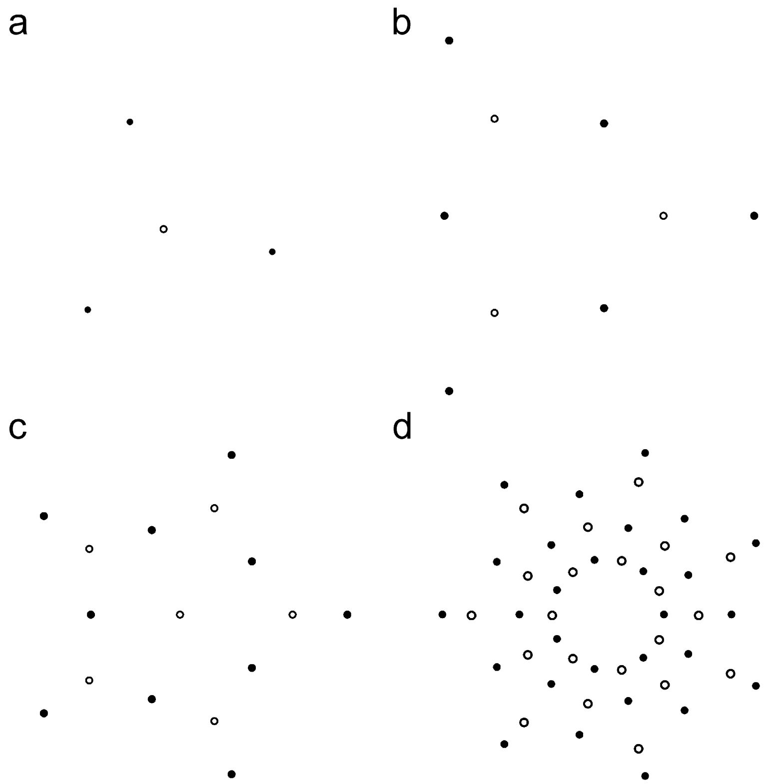

Fig. 5. Stationary equilibria generated from pairs of Adler–Moser polynomials. Majority–minority populations are (a) 3–1; (b) 6–3; (d) 10–6; (e) 28–21.

The polynomials generated by (31) are today known as the *Adler–Moser polynomials*. They arose in a study by Adler and Moser (1978) of the rational solutions of the Korteweg–de Vries equation. Bartman (1983) pointed out the connection to Tkachenko's equation.

There are explicit formulae for the polynomials P_n. These are given by Adler and Moser (1978), who acknowledge prior work by Crum (1955). Many of the results were also obtained even earlier in the aforementioned paper by Burchnall and Chaundy (1929). We simply state these formulae without proof. Consider the recursion

$$w_1 = z, \quad w_n'' = w_{n-1} \quad \text{for } n \geqslant 2.$$

This clearly leads to a sequence of polynomials,

$$w_2 = \frac{1}{3!}z^3 + az + b,$$
$$w_3 = \frac{1}{5!}z^5 + \frac{1}{3!}az^3 + \frac{1}{2!}bz^2 + cz + d,$$

and so on, where a, b, c, $d, \ldots$ are arbitrary constants.

Now consider the Wronskians

$$W_1(x_1) = w_1,$$
$$W_2(w_1, w_2) = \begin{vmatrix} w_1 & w_2 \\ w_1' & w_2' \end{vmatrix} = w_1 w_2' - w_1' w_2,$$
$$W_3(w_1, w_2, w_3) = \begin{vmatrix} w_1 & w_2 & w_3 \\ w_1' & w_2' & w_3' \\ w_1'' & w_2'' & w_3'' \end{vmatrix},$$

etc. The key insight is that the Adler–Moser polynomial P_n is proportional to the Wronskian W_n:

$$P_n = 1^n 3^{n-1} 5^{n-2} \cdots (2n-1)^1 \times W_n(w_1, w_2, ..., w_n). \tag{32}$$

We now have solutions of (29) for each n. Since (31) is a first order differential equation, an arbitrary constant is introduced in each integration. Thus, for a given configuration of the 'minority species' of vortices, there is a one-parameter family of polynomials whose roots give the locations of the 'majority species'. See also Kadtke and Campbell (1987).

Stationary patterns can now be displayed by determining the roots of the resulting polynomials. Since there are free parameters arising from the solution of the recursion relations, there are one-parameter families of solutions of the majority population given the positions of the minority population vortices. Some particularly symmetrical examples are shown in Fig. 5. See Aref et al. (2002) for further details.

This example shows the power of the 'method of polynomials' when it can be carried through: the generating polynomials, i.e., the polynomials with roots at the positions of the vortices, are obtained from the basic vortex equilibrium equations. Their roots then give the locations of the vortices.

The extension to uniformly translating configurations with equal numbers of positive and negative vortices is given in Aref et al. (2002). We shall not elaborate it here.

4. The flow field

Another instance in which the polynomial with roots at the location of the vortices has proven useful is in the discussion of the instantaneous flow field produced by a set of point vortices. Let us assume N vortices with strengths $\Gamma_1, \ldots, \Gamma_N$ are situated in the plane. For reasons that will be immediately obvious, we shall assume all the Γ to be rational. We assume that this is an adequate approximation for any set of given vortex strengths, which are generally thought of as real numbers. We shall assume, furthermore, that the Γs have been rescaled by multiplying each of them by the greatest common denominator for the entire set of rational numbers, i.e., that the Γs are all integers. Physically, this corresponds to changing the scales of length and/or time.

The two-dimensional flow field, (u, v), produced by these vortices is given by

$$u - \mathrm{i}v = \frac{1}{2\pi \mathrm{i}} \sum_{\alpha=1}^{N} \frac{\Gamma_\alpha}{z - z_\alpha}. \tag{33}$$

Now, consider the rational function

$$R(z) = (z - z_1)^{\Gamma_1} (z - z_2)^{\Gamma_2} \cdots (z - z_N)^{\Gamma_N}. \tag{34}$$

This is a rational function because of our convention that the Γs are integers. If all the Γs are positive, $R(z)$ is a polynomial.

At issue is the location of the zeros of the derivative of $R(z)$. We see that these zeros are the instantaneous stagnation points of the flow according to (33). We have, thus, a classical problem of analysis, viz to characterize the location of the zeros of a polynomial (or rational function) in terms of the zeros (and/or poles) of the polynomial (rational function) itself. The general, and very beautiful, result due to Siebeck (1864) is this: the stagnation points, i.e., zeros of (33), are the foci of a curve of class $N-1$ that touches each line segment connecting a z_α to a z_β in the point that divides the line segment in the ratio $\Gamma_\alpha : \Gamma_\beta$.

This result is proved early on in the book of Marden (1949). It has been elaborated and proved in a manner more accessible to fluid mechanicians by Aref and Brøns (1998).

Let us consider the simplest case of three vortices in some detail. The equation for a stagnation point is that the numerator on the right-hand side of Eq. (33) vanishes, i.e.,

$$\Gamma_1(z-z_2)(z-z_3)+\Gamma_2(z-z_1)(z-z_3)+\Gamma_3(z-z_1)(z-z_2)=0. \tag{35}$$

Unless $\Gamma_1+\Gamma_2+\Gamma_3=0$, in which case there is just one solution, this is a quadratic polynomial with two roots, $z=z_1^{(s)}$ and $z=z_2^{(s)}$. From (35) we get

$$z_1^{(s)}+z_2^{(s)}=z_1+z_2+z_3-\frac{\Gamma_1 z_1+\Gamma_2 z_2+\Gamma_3 z_3}{\Gamma_1+\Gamma_2+\Gamma_3}, \tag{36}$$

and

$$z_1^{(s)}z_2^{(s)}=\frac{\Gamma_1 z_2 z_3+\Gamma_2 z_3 z_1+\Gamma_3 z_1 z_2}{\Gamma_1+\Gamma_2+\Gamma_3}. \tag{37}$$

Choose coordinates such that $z_1=0$ and the x-axis is along the line connecting vortices 1 and 2, i.e., such that $z_2=c$ is real. Then $z_3=b\exp(\mathrm{i}A)$, where b is the length of the side connecting vortices 1 and 3, and A is the angle at vertex 1 of the vortex triangle. (For convenience think of vortices 1, 2 and 3 as being arranged counter-clockwise in the plane. The arguments, of course, can be adapted to either orientation of the vortex triangle.) Eq. (37) then simplifies considerably to

$$z_1^{(s)}z_2^{(s)}=\frac{\Gamma_1}{\Gamma_1+\Gamma_2+\Gamma_3}\,bc\exp(\mathrm{i}A). \tag{38}$$

This equation shows that if the angle between the line connecting vortex 1 (which is at the origin) to the stagnation point $z_1^{(s)}$ and the side of the triangle 12 is designated φ_1, and the corresponding angle for $z_2^{(s)}$ is designated φ_2, then $\varphi_1+\varphi_2=A$. This means that the angle, φ_1, between the ray from the origin to $z_1^{(s)}$ and the side 12 equals the angle, $A-\varphi_2$, between the ray from the origin to $z_2^{(s)}$ and the side 13. Pairs of points with this property are said to be *isogonal conjugate points*. It may be shown, cf. Coxeter (1992), that any pair of isogonal conjugate points in a triangle constitute the foci of a conic section inscribed in that triangle. This conic is the Siebeck conic. Using ideas and results from projective geometry one can find the explicit equation of this conic as detailed in Aref and Brøns (1998).

We illustrate the result by two examples reproduced from Fig. 1 of Aref and Brøns (1998). In Fig. 6(a) the symmetrical case of three identical vortices is illustrated. The vortex triangle is shown, with a vortex at each vertex. In this case the Siebeck conic is an ellipse. It touches the sides in their midpoints, the points that divide the side in the ratio 1:1. The two foci of the conic are the instanta-

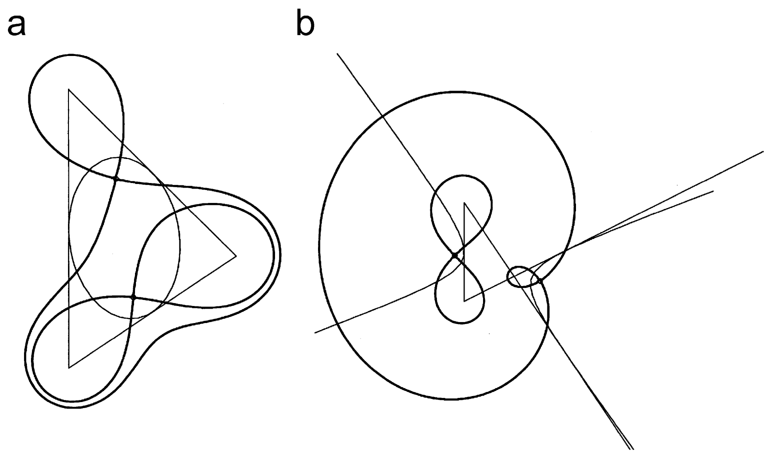

Fig. 6. Siebeck conic for vortex strengths (a) (1, 1, 1), (b) (1, 1, −1).

neous stagnation points. To amplify this we have drawn also the instantaneous dividing streamline which bifurcates at the two stagnation points.

In Fig. 6(b) we illustrate the case of three vortices with strengths $(1, 1, -1)$. Again, the vortex triangle is shown. This time the Siebeck conic is a hyperbola. It touches the side connecting the two identical vortices at its midpoint. It touches the two other sides in the points that divide those sides in the ratio $1 : (-1)$, i.e., in points located a side length further along on the extension of the side beyond the negative vortex. The foci of this hyperbola are the instantaneous stagnation points. To emphasize this, the instantaneous dividing streamline is shown. It has points of bifurcation at the stagnation points.

The problem discussed in this subsection is related to the general theme of finding the locations of the derivative of a polynomial given the location of its roots. The well-known *Gauss–Lucas* theorem states that the roots of the derivative of a polynomial are situated within the convex hull spanned by the roots of the polynomial itself. For positive vortices Siebeck's theorem provides a different restriction on the positions of the zeros (stagnation points). In the case of three vortices we saw that the stagnation points are the foci of an ellipse inscribed in the vortex triangle. This is a sharper characterization of their location than just knowing the stagnation points to be inside the vortex triangle.

We believe that further examples of the utility of generating functions that use the positions of vortices as their roots and/or poles will emerge, and hope that the examples presented in this paper may induce others to seek out further instances where 'the method of polynomials' is useful and productive.

We would like to believe that this interplay between model problems in fluid mechanics and basic mathematical structures, such as polynomials, rational functions and their singularities would have appealed to Professor Imai.

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This paper is dedicated to the memory of Professor Isao Imai.

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